Generalized Jaynes–Cummings Hamiltonians by shape-invariant hierarchies and their SUSY partners

2006 J. Phys. A: Math. Gen. 39 11301

(http://iopscience.iop.org/0305-4470/39/36/011)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 11301–11311 doi:10.1088/0305-4470/39/36/011

Generalized Jaynes–Cummings Hamiltonians by shape-invariant hierarchies and their SUSY partners

V Hussin[1], **Ş Kuru**[2,3] **and J Negro**[2]

[1] Centre de recherches mathématiques et Département de mathématiques et de statistique, Université de Montréal, C P 6128, succ. Centre-ville, Montréal (Québec), H3C 3J7 Canada
[2] Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071 Valladolid, Spain

Received 9 June 2006, in final form 18 July 2006
Published 18 August 2006
Online at stacks.iop.org/JPhysA/39/11301

Abstract

A generalization of the matrix Jaynes–Cummings model in the rotating wave approximation is proposed by means of the shape-invariant hierarchies of scalar factorized Hamiltonians. A class of Darboux transformations (sometimes called SUSY transformations in this context) suitable for these generalized Jaynes–Cummings models is constructed. Finally one example is worked out using the methods developed.

PACS numbers: 03.65.−w, 11.30.Pb, 42.50.−p

1. Introduction

The Darboux transformation is a very useful technique in areas like supersymmetric (SUSY) quantum mechanics [1] and nonlinear equations [2]. In particular, there are many works in the literature concerning the one-dimensional problems in quantum mechanics for scalar wavefunctions [1, 3–5]. However, it is not so well studied in the context of matrix wave equations of Schrödinger type. Some developments of this theory have been made [6–8], but it is not easy to find good examples in the matrix case since, once a known initial Hamiltonian is fixed, the hermiticity of the partner Hamiltonian is a property difficult to guarantee.

One interesting physical example was given for a Pauli equation of a spin-$1/2$ neutral particle, where it was shown that for the radial equation the Darboux transformations lead to matrix shape invariance as well as to many other algebraic properties (see [9] and references therein).

Another physical case of a matrix Hamiltonian of Schrödinger type is the Jaynes–Cummings (J-C) Hamiltonian describing the interaction between a two-level atom and a quantized single-mode electromagnetic field, in the rotating wave approximation [10]. There

[3] On leave of absence from Department of Physics, Faculty of Science, Ankara University, 06100 Ankara, Turkey.

have been many studies of this model (see for instance [11]) as well as many variations [12–14]. An application of Darboux transformations to the J-C model was presented in [15].

In this work we want to define a new class of J-C Hamiltonians with the help of shape-invariant scalar Hamiltonian hierarchies [16] in the line of [17]. In this way, we propose to show that many particular properties previously worked out in [15] have a simple explanation and also a more general character. We have also tried to make a self-contained work easy to follow by any reader familiar with the techniques of SUSY quantum mechanics.

The organization of the paper is as follows. We first introduce, in section 2, a new class of generalized J-C Hamiltonians and we study its spectrum. In section 3, we determine the matrix Darboux transformations suitable for this class of Hamiltonians, and examine three cases, leading to Hermitian SUSY partner Hamiltonians. The Darboux transformations of second order are also considered. In section 4, these transformations are applied to the generalized J-C Hamiltonian obtained from the radial oscillator hierarchy. Finally, we end the paper with some conclusions and remarks.

2. Generalized J-C Hamiltonians

We start from a class of factorized shape-invariant one-dimensional Hamiltonian hierarchy $\{H_\ell\}$, $\ell \in \mathbb{Z}$, in the sense given for instance in [3, 16]. This hierarchy is obtained with the help of a sequence of first-order differential operators

$$A_\ell^\pm = \mp\partial_x + f(x,\ell), \tag{1}$$

where $\partial_x = \mathrm{d}/\mathrm{d}x$, and $f(x,\ell)$ are certain functions, with the conditions

$$H_\ell = A_\ell^+ A_\ell^- + \lambda_\ell = A_{\ell-1}^- A_{\ell-1}^+ + \lambda_{\ell-1}, \tag{2}$$

with λ_ℓ being constants depending only on ℓ. The equation

$$A_\ell^- \psi_\ell^0 = 0 \tag{3}$$

determines a square-integrable wavefunction that is the ground state of H_ℓ with energy $E_\ell^0 = \lambda_\ell$. The excited states of H_ℓ with energy $E_\ell^n = \lambda_{\ell+n}$, $n = 0, 1, 2, \ldots,$ are obtained by means of the raising operators

$$\psi_\ell^n = A_\ell^+ A_{\ell+1}^+ \ldots A_{\ell+n-1}^+ \psi_{\ell+n}^0. \tag{4}$$

This Hamiltonian hierarchy satisfies the intertwining relations

$$A_\ell^- H_\ell = H_{\ell+1} A_\ell^-, \qquad A_\ell^+ H_{\ell+1} = H_\ell A_\ell^+. \tag{5}$$

This means that

$$A_\ell^- \psi_\ell^n \propto \psi_{\ell+1}^{n-1}, \qquad A_\ell^+ \psi_{\ell+1}^{n-1} \propto \psi_\ell^n, \tag{6}$$

where we can easily find the proportionality factors. Indeed, we have

$$\langle A_\ell^- \psi_\ell^n, A_\ell^- \psi_\ell^n\rangle = \langle \psi_\ell^n, A_\ell^+ A_\ell^- \psi_\ell^n\rangle = (\lambda_{\ell+n} - \lambda_\ell)\langle \psi_\ell^n, \psi_\ell^n\rangle, \tag{7}$$

and, in a similar way,

$$\langle A_\ell^+ \psi_{\ell+1}^{n-1}, A_\ell^+ \psi_{\ell+1}^{n-1}\rangle = (\lambda_{\ell+n} - \lambda_\ell)\langle \psi_{\ell+1}^{n-1}, \psi_{\ell+1}^{n-1}\rangle. \tag{8}$$

Therefore we can write, up to a phase, the action of the lowering and raising operators on the physical square-integrable eigenfunctions

$$A_\ell^+ \psi_{\ell+1}^{n-1} = \sqrt{\lambda_{\ell+n} - \lambda_\ell}\,\psi_\ell^n, \qquad A_\ell^- \psi_\ell^n = \sqrt{\lambda_{\ell+n} - \lambda_\ell}\,\psi_{\ell+1}^{n-1}. \tag{9}$$

By making use of the above hierarchy, we propose the following J-C type Hamiltonian:

$$H = \begin{pmatrix} H_{\ell+1} + \alpha & \beta A_\ell^- \\ \beta A_\ell^+ & H_\ell \end{pmatrix}, \tag{10}$$

which can also be rewritten as

$$H = \left(A_\ell^+ A_\ell^- + \lambda_\ell\right)\sigma_0 + \tfrac{1}{2}\left(\left[A_\ell^-, A_\ell^+\right] + \alpha\right)(\sigma_3 + 1) + \beta\left(A_\ell^- \sigma_+ + A_\ell^+ \sigma_-\right), \tag{11}$$

where α and β are real constants, σ_0 is the identity matrix, $\sigma_i, i = 1, 2, 3$, are the Pauli matrices, and $\sigma_\pm = (\sigma_1 \pm \mathrm{i}\sigma_2)/2$. This model is quite similar to that studied in [17]. Now we can compute explicitly the eigenvalues $\mathcal{E}$ and the two-component eigenfunctions Ψ of this matrix Hamiltonian. Next we will consider three sets of eigenfunctions.

2.1. Physical eigenfunctions

As the Hamiltonian H commutes with the operator

$$\mathcal{N} = \begin{pmatrix} H_{\ell+1} & 0 \\ 0 & H_\ell \end{pmatrix}, \tag{12}$$

let us propose a common eigenfunction of both H and $\mathcal{N}$ of the form

$$\Psi_n = \begin{pmatrix} c_1 \psi_{\ell+1}^{n-1} \\ c_2 \psi_\ell^n \end{pmatrix} \qquad n = 1, 2 \ldots. \tag{13}$$

The constants c_1, c_2 are to be determined from the equation

$$H\Psi_n = \mathcal{E}_n \Psi_n, \tag{14}$$

where $\mathcal{E}_n$ is also unknown. Using (10) and (13) in (14) together with (9), we obtain the matrix equation

$$\begin{pmatrix} \lambda_{\ell+n} + \alpha & \beta\sqrt{\lambda_{\ell+n} - \lambda_\ell} \\ \beta\sqrt{\lambda_{\ell+n} - \lambda_\ell} & \lambda_{\ell+n} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathcal{E}_n \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{15}$$

We thus get the eigenvalue solutions of our generalized J-C Hamiltonian

$$\mathcal{E}_n^\pm = \tfrac{1}{2}\left(\alpha + 2\lambda_{\ell+n} \pm \sqrt{\alpha^2 + 4\beta^2(\lambda_{\ell+n} - \lambda_\ell)}\right) \tag{16}$$

and the two sets of eigenfunctions

$$\Psi_n^\pm = \begin{pmatrix} c_1^\pm \psi_{\ell+1}^{n-1} \\ c_2^\pm \psi_\ell^n \end{pmatrix} \tag{17}$$

such that the coefficients $c_1^\pm$ and $c_2^\pm$ are related by

$$c_1^\pm = \frac{\mathcal{E}_n^\pm - \lambda_{\ell+n}}{\beta\sqrt{\lambda_{\ell+n} - \lambda_\ell}}\, c_2^\pm. \tag{18}$$

2.2. Non-physical eigenfunctions

We can obtain general non-physical eigenfunctions by a slight modification of the above procedure that will be very useful to apply Darboux transformations in the following sections. Therefore let us consider a non-physical eigenfunction ψ_ℓ^ϵ of H_ℓ with a real eigenvalue ϵ (different from $\lambda_{\ell+n}$):

$$H_\ell \psi_\ell^\epsilon = \epsilon \psi_\ell^\epsilon. \tag{19}$$

Now we propose the matrix eigenfunction of the form

$$\Psi_\epsilon = \begin{pmatrix} c_1 A_\ell^- \psi_\ell^\epsilon \\ c_2 \psi_\ell^\epsilon \end{pmatrix}, \tag{20}$$

where $A_\ell^- \psi_\ell^\epsilon \neq 0$, which satisfies the eigenvalue equation

$$H\Psi_\epsilon = \mathcal{E}_\epsilon \Psi_\epsilon. \tag{21}$$

After similar computations as in the previous case, we are led to the eigenvalues

$$\mathcal{E}_\epsilon^\pm = \tfrac{1}{2}\left(\alpha + 2\epsilon \pm \sqrt{\alpha^2 + 4\beta^2(\epsilon - \lambda_\ell)}\right). \tag{22}$$

The coefficients of the corresponding eigenfunctions $\Psi_\epsilon^\pm$ are related by

$$c_1^\pm = \frac{\mathcal{E}_\epsilon^\pm - \epsilon}{\beta(\epsilon - \lambda_\ell)} c_2^\pm. \tag{23}$$

Other expressions of non-physical eigenfunctions can be obtained by choosing the form

$$\Psi_\epsilon = \begin{pmatrix} c_1 \psi_{\ell+1}^\epsilon \\ c_2 A_\ell^+ \psi_{\ell+1}^\epsilon \end{pmatrix}, \tag{24}$$

where $H_{\ell+1}\psi_{\ell+1}^\epsilon = \epsilon \psi_{\ell+1}^\epsilon$ and $A_\ell^+ \psi_{\ell+1}^\epsilon \neq 0$.

2.3. Fundamental eigenfunctions

Taking into account the definition of the ground state (3), we can express the functions $f(x,\ell)$ in (1) as

$$f(x,\ell) = -\left(\psi_\ell^0\right)_x \big/ \psi_\ell^0, \tag{25}$$

where henceforth we use a subindex x, as in $\left(\psi_\ell^0\right)_x$, to express the derivative with respect to x. We can construct from ψ_ℓ^0 the physical fundamental two-component eigenfunction

$$\Psi_0^- = \begin{pmatrix} 0 \\ \psi_\ell^0 \end{pmatrix}, \tag{26}$$

satisfying

$$H\Psi_0^- = \lambda_\ell \Psi_0^-. \tag{27}$$

In the same way, since

$$A_\ell^+ \psi_{\ell+1}^{-1} = 0, \tag{28}$$

where $\psi_{\ell+1}^{-1}$ is a non-physical eigenfunction of $H_{\ell+1}$, we have another expression for $f(x,\ell)$:

$$f(x,\ell) = \left(\psi_{\ell+1}^{-1}\right)_x \big/ \psi_{\ell+1}^{-1}. \tag{29}$$

Then, we can build a second (non-physical) fundamental eigenfunction

$$\Psi_0^+ = \begin{pmatrix} \psi_{\ell+1}^{-1} \\ 0 \end{pmatrix}, \tag{30}$$

satisfying

$$H\Psi_0^+ = (\lambda_\ell + \alpha)\Psi_0^+. \tag{31}$$

These two eigenfunctions and their eigenvalues can be considered as the limit of (15)–(18) when $n = 0$.

3. Darboux transformations for generalized J-C Hamiltonians

3.1. Matrix Darboux transformations

By replacing the expressions of the lowering and raising operators (1) and using (2), in the J-C Hamiltonian (10), we get a second-order matrix differential operator of the form

$$H = -\partial_x^2 + V(x) + \beta\gamma\partial_x, \tag{32}$$

where

$$V(x)=\begin{pmatrix} f(x,\ell)^2 + f_x(x,\ell) + \lambda_\ell + \alpha & \beta f(x,\ell) \\ \beta f(x,\ell) & f(x,\ell)^2 - f_x(x,\ell) + \lambda_\ell \end{pmatrix}, \qquad \gamma=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{33}$$

Now we look for a matrix Darboux transformation

$$L = \partial_x - \mathbb{W}(x), \tag{34}$$

$\mathbb{W}(x)$ being a matrix-valued function, usually called superpotential, satisfying the relation

$$LH = \widetilde{H}L, \tag{35}$$

where

$$\widetilde{H} = -\partial_x^2 + \widetilde{V}(x) + \widetilde{\beta}\gamma\partial_x \tag{36}$$

is another Hamiltonian of the same type as (32). Given any pair of eigenfunctions Ψ_1, Ψ_2 of H with eigenvalues $\mathcal{E}_1, \mathcal{E}_2$, we can build the 2×2 matrix

$$\mathbb{M}(x) = (\Psi_1, \Psi_2) \tag{37}$$

such that

$$H\mathbb{M}(x) = \mathbb{M}(x)\mathbb{E}, \tag{38}$$

where $\mathbb{E} = \mathrm{diag}(\mathcal{E}_1, \mathcal{E}_2)$. Then the intertwining relation (35) implies that $\widetilde{\mathbb{M}}(x) = L\mathbb{M}(x)$ will be another matrix of solutions of $\widetilde{H}$ with the same eigenvalue matrix $\mathbb{E}$:

$$\widetilde{H}\widetilde{\mathbb{M}}(x) = \widetilde{\mathbb{M}}(x)\mathbb{E}. \tag{39}$$

Let us choose these eigenfunctions Ψ_1, Ψ_2 belonging to the kernel of L, i.e.,

$$L\mathbb{M}(x) = 0. \tag{40}$$

Then, it is well known [7, 8, 15] that, from the intertwining relation (35), also their images under H will belong to that kernel, i.e., $L(H\mathbb{M}(x)) = 0$. Therefore, from (40) and (34) it can be easily seen that $\mathbb{W}(x)$ is given by

$$\mathbb{W}(x) = \mathbb{M}_x(x)\mathbb{M}(x)^{-1}. \tag{41}$$

Now we can expand the intertwining relation (35) taking into account expressions (32), (36) and (34), by equating the powers of derivatives in x to get the following results.

- Second power terms:

$$\beta = \widetilde{\beta}. \tag{42}$$

- First power terms:

$$\widetilde{V}(x) = V(x) - 2\mathbb{W}_x(x) + \beta[\gamma, \mathbb{W}(x)]. \tag{43}$$

- Zero power terms:

$$\mathbb{W}_{xx}(x) + 2\mathbb{W}_x(x)\mathbb{W}(x) + [\mathbb{W}(x), V(x)] - V_x(x) - \beta([\gamma, \mathbb{W}(x)]\mathbb{W}(x) + \gamma\mathbb{W}_x(x)) = 0. \tag{44}$$

The last relation is a consistency condition for the superpotential $\mathbb{W}(x)$. However, if we substitute expression (41) in (44), using (32) and (38), we can check that this relation is automatically satisfied. Hence any first-order operator L in the form (34) with $\mathbb{W}(x)$ given by (41) is a Darboux transformation.

3.2. Special superpotentials for J-C Hamiltonians

3.2.1. Superpotential associated with the pair of fundamental eigenfunctions. In this case, we form the matrix $\mathbb{M}_0$ with the simplest solutions $\Psi_0^{\pm}$, so that

$$\mathbb{M}_0(x) = \begin{pmatrix} \psi_{\ell+1}^{-1} & 0 \\ 0 & \psi_\ell^0 \end{pmatrix}. \tag{45}$$

Replacing in (41) and taking into account (25) and (29) we get the superpotential

$$\mathbb{W}_0(x) = \begin{pmatrix} \left(\psi_{\ell+1}^{-1}\right)_x/\psi_{\ell+1}^{-1} & 0 \\ 0 & \left(\psi_\ell^0\right)_x/\psi_\ell^0 \end{pmatrix} = \begin{pmatrix} f(x,\ell) & 0 \\ 0 & -f(x,\ell) \end{pmatrix}, \tag{46}$$

and the Darboux operator L given in (34) takes the simple form

$$L_0 = \begin{pmatrix} -A_\ell^+ & 0 \\ 0 & A_\ell^- \end{pmatrix}. \tag{47}$$

The partner Hamiltonian is thus given by

$$\widetilde{H} = \begin{pmatrix} H_\ell + \alpha & -\beta A_\ell^+ \\ -\beta A_\ell^- & H_{\ell+1} \end{pmatrix}. \tag{48}$$

Note that in this case both the initial and partner Hamiltonians can also be expressed in terms of the matrix Darboux operators L_0 and its conjugate $L_0^\dagger$ as

$$H = L_0^\dagger L_0 + \beta\gamma L_0 + \lambda_\ell\sigma_0 + \tfrac{1}{2}\alpha(\sigma_3 + 1), \tag{49}$$

$$\widetilde{H} = L_0 L_0^\dagger + \beta L_0\gamma + \lambda_\ell\sigma_0 + \tfrac{1}{2}\alpha(\sigma_3 + 1). \tag{50}$$

Since $L_0^\dagger L_0$ and $L_0 L_0^\dagger$ are symmetries, i.e., they commute with H and $\widetilde{H}$, respectively, S and $\widetilde{S}$ defined by

$$S = \beta\gamma L_0 + \lambda_\ell\sigma_0 + \tfrac{1}{2}\alpha(\sigma_3 + 1), \qquad \widetilde{S} = \beta L_0\gamma + \lambda_\ell\sigma_0 + \tfrac{1}{2}\alpha(\sigma_3 + 1) \tag{51}$$

are also first-order Hermitian symmetries of H and $\widetilde{H}$ (a particular case of these symmetries is given in [17]).

3.2.2. Superpotentials associated with a pair of $\mathcal{E}^{\pm}$-eigenfunctions. Let us choose two physical eigenfunctions in the specific form (17) with the eigenvalues $\mathcal{E}_n^{\pm}$ as given in (16). Then the 2×2 matrix $\mathbb{M}_n(x)$ takes the form

$$\mathbb{M}_n(x) = \begin{pmatrix} c_1^+\psi_{\ell+1}^{n-1} & c_1^-\psi_{\ell+1}^{n-1} \\ c_2^+\psi_\ell^n & c_2^-\psi_\ell^n \end{pmatrix} = \mathbb{D}_n(x)\mathbb{R}_n, \tag{52}$$

where

$$\mathbb{D}_n(x) = \begin{pmatrix} \psi_{\ell+1}^{n-1} & 0 \\ 0 & \psi_\ell^n \end{pmatrix}, \qquad \mathbb{R}_n = \begin{pmatrix} c_1^+ & c_1^- \\ c_2^+ & c_2^- \end{pmatrix}. \tag{53}$$

According to formula (41), the superpotential $\mathbb{W}_n(x)$ can be written as

$$\mathbb{W}_n(x) = (\mathbb{D}_n(x)\mathbb{R}_n)_x(\mathbb{D}_n(x)\mathbb{R}_n)^{-1} = \mathbb{D}_{nx}(x)\mathbb{D}_n(x)^{-1}. \tag{54}$$

Hence, we get the final explicit expression for the superpotential

$$\mathbb{W}_n(x) = \begin{pmatrix} \left(\psi_{\ell+1}^{n-1}\right)_x/\psi_{\ell+1}^{n-1} & 0 \\ 0 & \left(\psi_\ell^n\right)_x/\psi_\ell^n \end{pmatrix}. \tag{55}$$

Therefore, by varying n ($\neq 0$) we can produce many diagonal superpotentials. In the same way, we can make use of other pairs of non-physical eigenfunctions $\Psi_\epsilon^\pm$ as they were given in the previous section leading to a similar diagonal expression as (55). This is quite important since we can choose non-physical eigenfunctions, without zeros, in order to avoid singularities of the partner potential.

However, if we choose any other combination of eigenfunctions of H we will get a superpotential that in general will not be diagonal nor Hermitian, leading to a non-Hermitian partner Hamiltonian. But, there are some exceptions as we will see in the following.

3.2.3. Superpotentials associated with triangular matrix solutions. Another option that allows us to get reasonable simple superpotentials, leading to Hermitian potentials, is obtained if we choose a triangular matrix solution for $\mathbb{M}(x)$, i.e.,

$$\mathbb{M}(x) = \begin{pmatrix} 0 & c_1^\pm \psi_{\ell+1}^{n-1} \\ \psi_\ell^0 & c_2^\pm \psi_\ell^n \end{pmatrix}. \tag{56}$$

Substituting in (41) we also get a triangular superpotential

$$\mathbb{W}(x) = \begin{pmatrix} \left(\psi_{\ell+1}^{n-1}\right)_x / \psi_{\ell+1}^{n-1} & 0 \\ -\frac{c_2^\pm}{c_1^\pm} \frac{(\psi_\ell^0)_x \psi_\ell^n - (\psi_\ell^n)_x \psi_\ell^0}{\psi_\ell^0 \psi_{\ell+1}^{n-1}} & \left(\psi_\ell^0\right)_x / \psi_\ell^0 \end{pmatrix}. \tag{57}$$

The terms with derivatives in the non-diagonal element of this matrix can be substituted as follows. By using the differential expression (1) of A_ℓ^-, we get from (3)

$$\left(\psi_\ell^0\right)_x = -f(x,\ell)\psi_\ell^0 \tag{58}$$

and from (9)

$$\left(\psi_\ell^n\right)_x = \sqrt{\lambda_{n+\ell} - \lambda_\ell}\,\psi_{\ell+1}^{n-1} - f(x,\ell)\psi_\ell^n. \tag{59}$$

Therefore, having in mind (18), the final expression for the $\mathbb{W}(x)$ matrix is

$$\mathbb{W}(x) = \begin{pmatrix} \left(\psi_{\ell+1}^{n-1}\right)_x / \psi_{\ell+1}^{n-1} & 0 \\ \frac{\beta(\lambda_{\ell+n} - \lambda_\ell)}{\mathcal{E}_n^\pm - \lambda_{\ell+n}} & \left(\psi_\ell^0\right)_x / \psi_\ell^0 \end{pmatrix}. \tag{60}$$

We must point out that when we use triangular matrices with non-physical solutions the constant non-diagonal term in $\mathbb{W}(x)$ must be changed according to (23).

3.2.4. Superpotentials of second-order Darboux transformations. We can also build diagonal second-order matrix Darboux transformations by using two matrices $\mathbb{M}_n(x)$ and $\mathbb{M}_m(x)$ with the two sets of eigenfunctions $\Psi_n^\pm$ and $\Psi_m^\pm$. Then the second-order Darboux transformation $L^{(2)}$ associated with these matrices is such that

$$L^{(2)}\mathbb{M}_n = L^{(2)}\mathbb{M}_m = 0. \tag{61}$$

Therefore, we can write it as

$$L^{(2)} = L_2 L_1 = (\partial_x - \widetilde{\mathbb{W}}_m)(\partial_x - \mathbb{W}_n) \tag{62}$$

with

$$L_1 \mathbb{M}_n(x) = 0, \qquad L_2 \widetilde{\mathbb{M}}_m(x) = 0, \tag{63}$$

where

$$\widetilde{\mathbb{M}}_m(x) = L_1 \mathbb{M}_m(x) = (\mathbb{D}_{m\,x}(x) - \mathbb{W}_n(x)\mathbb{D}_m(x))\mathbb{R}_m. \tag{64}$$

Hence, $\mathbb{W}_n(x)$ is given by formula (41), while

$$\widetilde{\mathbb{W}}_m(x) = (\mathbb{D}_{mx}(x) - \mathbb{W}_n(x)\mathbb{D}_m(x))_x\left(\mathbb{D}_{mx}(x) - \mathbb{W}_n(x)\mathbb{D}_m(x)\right)^{-1}$$
$$= \begin{pmatrix} \log_x \frac{W\left(\psi_{\ell+1}^{n-1}, \psi_{\ell+1}^{m-1}\right)}{\psi_{\ell+1}^{n-1}} & 0 \\ 0 & \log_x \frac{W\left(\psi_\ell^n, \psi_\ell^m\right)}{\psi_\ell^n} \end{pmatrix}, \tag{65}$$

$W(\psi, \phi)$ being the Wronskian of two functions. Now, the second-order intertwining is

$$L^{(2)}H = L_2L_1H = L_2\widetilde{H}L_1 = \widetilde{\widetilde{H}}L_2L_1 = \widetilde{\widetilde{H}}L^{(2)}. \tag{66}$$

The action of L_2 on $\widetilde{H}$, whose potential is given in (43), leads to the partner potential

$$\widetilde{\widetilde{V}}(x) = \widetilde{V}(x) - 2\widetilde{\mathbb{W}}_{mx}(x) + \beta[\gamma, \widetilde{\mathbb{W}}_m(x)]. \tag{67}$$

Taking into account the expression of $\widetilde{V}$, we arrive at

$$\widetilde{\widetilde{V}}(x) = V(x) - 2\widetilde{\widetilde{\mathbb{W}}}_x(x) + \beta[\gamma, \widetilde{\widetilde{\mathbb{W}}}(x)], \tag{68}$$

with

$$\widetilde{\widetilde{\mathbb{W}}}(x) = \mathbb{W}_n(x) + \widetilde{\mathbb{W}}_m(x) = \begin{pmatrix} \log_x W\left(\psi_{\ell+1}^{n-1}, \psi_{\ell+1}^{m-1}\right) & 0 \\ 0 & \log_x W\left(\psi_\ell^n, \psi_\ell^m\right) \end{pmatrix}. \tag{69}$$

The final Hamiltonian is given by

$$\widetilde{\widetilde{H}} = -\partial_x^2 + \widetilde{\widetilde{V}}(x) + \beta\gamma\partial_x. \tag{70}$$

4. Example: radial oscillator like J-C Hamiltonians

In this section we shall consider as an example the J-C Hamiltonians obtained by means of the radial oscillator hierarchy given by the Hamiltonians [18]

$$H_\ell = -\partial_r^2 + \frac{\ell(\ell-1)}{r^2} + r^2 + 2\ell. \tag{71}$$

In this case the lowering and raising operators of (1) are given by

$$A_\ell^\pm = \mp\partial_r - \frac{\ell}{r} + r \tag{72}$$

and relation (2) is fulfilled with $\lambda_\ell = 4\ell + 1$. The general eigenfunctions for any eigenvalue ϵ are given by [19]

$$\psi_\ell^\epsilon(r) = r^\ell\, \mathrm{e}^{-r^2/2}(C_1\, {}_1F_1((1-\epsilon)/4 + \ell, \ell + 1/2; r^2)$$
$$+ C_2 r^{-2\ell+1}\, {}_1F_1((3-\epsilon)/4, -\ell + 3/2; r^2)). \tag{73}$$

The physical eigenfunctions of (71) are obtained when $C_2 = 0$ and the remaining hypergeometric function is a polynomial. That is, the first argument of the function ${}_1F_1$ must be a negative integer, $(1-\epsilon)/4 + \ell = -n$, $n = 0, 1, 2 \ldots$. Therefore, these eigenfunctions are

$$\psi_\ell^n(r) = Cr^\ell\, \mathrm{e}^{-r^2/2}\, {}_1F_1(-n, \ell + 1/2; r^2), \tag{74}$$

where C is a normalization constant and the corresponding discrete energy eigenvalues are

$$E_\ell^n = 4(\ell + n) + 1. \tag{75}$$

The matrix J-C potential corresponding to this hierarchy is

$$V(r) = \begin{pmatrix} \frac{\ell(\ell+1)}{r^2} + r^2 + 2(\ell+1) + \alpha & \beta\left(-\frac{\ell}{r} + r\right) \\ \beta\left(-\frac{\ell}{r} + r\right) & \frac{\ell(\ell-1)}{r^2} + r^2 + 2\ell \end{pmatrix}. \tag{76}$$

The physical eigenfunctions of the J-C Hamiltonian with potential (76) obtained by means of (74) will be called $\Psi_n^\pm$.

4.1. Superpotential of fundamental solutions

The so-called fundamental eigenfunctions obtained from (3) and (28) are here

$$\psi_\ell^0(r) = r^\ell\, \mathrm{e}^{-r^2/2}, \qquad \psi_{\ell+1}^{-1}(r) = r^{-\ell}\, \mathrm{e}^{r^2/2}. \tag{77}$$

From these solutions, by using (46), we get the explicit superpotential

$$\mathbb{W}_0(r) = \begin{pmatrix} -\frac{\ell}{r} + r & 0 \\ 0 & \frac{\ell}{r} - r \end{pmatrix}. \tag{78}$$

The new potential, in agreement with (48), is given by

$$\widetilde{V}(r) = \begin{pmatrix} \frac{\ell(\ell-1)}{r^2} + r^2 + 2\ell + \alpha & \beta\left(\frac{\ell}{r} - r\right) \\ \beta\left(\frac{\ell}{r} - r\right) & \frac{\ell(\ell+1)}{r^2} + r^2 + 2(\ell+1) \end{pmatrix}. \tag{79}$$

4.2. Superpotential associated with a pair of non-physical solutions

Now, in order to avoid singularities, we will choose the special non-physical eigenfunctions in the form of (74), replacing $-n$ by n. We use the notation $\Psi_{-n}^{\pm}, n = 1, 2, 3, \ldots,$ for such non-physical eigenfunctions. Therefore, from (55) after some computations we get the superpotential [20]

$$\mathbb{W}_{-n}(r) = \begin{pmatrix} \frac{\ell+1}{r} - r + h_1(r) & 0 \\ 0 & \frac{\ell}{r} - r + h_2(r) \end{pmatrix}, \tag{80}$$

where $h_1(r) = \log_r {}_1F_1(n+1, l+3/2; r^2)$, and $h_2(r) = \log_r {}_1F_1(n, l+1/2; r^2)$. Then, after substituting (80) in (43), using the initial potential (76) we get the partner potential

$$\widetilde{V}(r) = \begin{pmatrix} \frac{(\ell+1)(\ell+2)}{r^2} + r^2 + 2(\ell+2) + \alpha - 2h_{1r}(r) & \beta\left(-\frac{\ell+1}{r} + r + h_2(r) - h_1(r)\right) \\ \beta\left(-\frac{\ell+1}{r} + r + h_2(r) - h_1(r)\right) & \frac{\ell(\ell+1)}{r^2} + r^2 + 2(\ell+1) - 2h_{2r}(r) \end{pmatrix}. \tag{81}$$

The corresponding partner Hamiltonian (36) with $\widetilde{\beta} = \beta$, when we substitute (81), has a simple expression,

$$\widetilde{H}(r) = \begin{pmatrix} H_{\ell+2} + \alpha - 2h_{1r}(r) & \beta(A_{\ell+1}^{-} + h_2(r) - h_1(r)) \\ \beta\left(A_{\ell+1}^{+} + h_2(r) - h_1(r)\right) & H_{\ell+1} - 2h_{2r}(r) \end{pmatrix}. \tag{82}$$

4.3. Superpotential associated with triangular solutions

Finally, we shall take the triangular matrix (56) of the physical eigenfunctions with the solutions given by (74), to get the superpotential

$$\mathbb{W}^{\pm}(r) = \begin{pmatrix} \frac{\ell+1}{r} - r - h(r) & 0 \\ K^{\pm} & \frac{\ell}{r} - r \end{pmatrix}, \tag{83}$$

where [20]

$$K^{\pm} = \frac{8\beta n}{\alpha \pm \sqrt{\alpha^2 + 16\beta^2 n}}, \qquad h(r) = -\log_r {}_1F_1(1-n, l+3/2; r^2). \tag{84}$$

Then, after substituting (83) in (43), with the initial potential (76), we get the partner potential

$$\widetilde{V}^{\pm}(r) = \begin{pmatrix} \frac{(\ell+1)(\ell+2)}{r^2} + r^2 + 2(\ell+2) + \alpha + 2h_r(r) + \beta K^{\pm} & \beta\left(-\frac{\ell+1}{r} + r + h(r)\right) \\ \beta\left(-\frac{\ell+1}{r} + r + h(r)\right) & \frac{\ell(\ell+1)}{r^2} + r^2 + 2(\ell+1) - \beta K^{\pm} \end{pmatrix}. \tag{85}$$

The corresponding partner Hamiltonian is

$$\widetilde{H}^{\pm}(r) = \begin{pmatrix} H_{\ell+2} + \alpha + 2h_r(r) + \beta K^{\pm} & \beta(A^{-}_{\ell+1} + h(r)) \\ \beta\left(A^{+}_{\ell+1} + h(r)\right) & H_{\ell+1} - \beta K^{\pm} \end{pmatrix}. \tag{86}$$

In order to avoid singularities in this partner Hamiltonian we should take non-physical solutions choosing for example the values $n = -1, -2, -3\ldots,$ as in the previous subsection.

4.4. Superpotential and Hamiltonians associated with second-order Darboux transformations

We will compute the second-order superpotential choosing the particular matrix solutions $\mathbb{M}_n(r)$ with $n = 1$ and $\mathbb{M}_m(r)$ with $m = 2$. After some computations the matrix superpotential (69) becomes

$$\approx{\mathbb{W}}(r) = \begin{pmatrix} \frac{3+2\ell}{r} - 2r & 0 \\ 0 & \frac{1+2\ell}{r} - 2r + g(r) \end{pmatrix}, \tag{87}$$

where

$$g(r) = -\frac{8r(1 + 2\ell - 2r^2)}{3 + 4\ell(2 + \ell) - 4r^2(1 + 2\ell - r^2)}. \tag{88}$$

Then using formula (68) we get the partner potential

$$\overset{\approx}{V}(r) = \begin{pmatrix} \frac{(\ell+2)(\ell+3)}{r^2} + r^2 + 2(\ell + 3) + \alpha & \beta\left(-\frac{\ell+1}{r} + r + g(r)\right) \\ \beta\left(-\frac{\ell+1}{r} + r + g(r)\right) & \frac{(\ell+1)(\ell+2)}{r^2} + r^2 + 2(\ell + 2) - 2g_r(r) \end{pmatrix}, \tag{89}$$

so that the corresponding partner Hamiltonian takes the form

$$\overset{\approx}{H}(r) = \begin{pmatrix} H_{\ell+3} + \alpha & \beta(A^{-}_{\ell+1} + g(r)) \\ \beta\left(A^{+}_{\ell+1} + g(r)\right) & H_{\ell+2} - 2g_r(r) \end{pmatrix}. \tag{90}$$

Although the first-order partner Hamiltonian with physical eigenfunctions is singular, this second-order partner Hamiltonian does not have any singularity but its spectrum differs from the initial J-C Hamiltonian in that it has four levels less: those with eigenvalues $\mathcal{E}_1^{\pm}, \mathcal{E}_2^{\pm}$ (see also [21]).

5. Conclusions

In this work we have extended some methods previously used in [15] to a class of generalized J-C Hamiltonians. In doing so we have shown some interesting properties. (i) A class of superpotentials automatically takes a diagonal form, leading to non-trivial SUSY partner Hamiltonians. The same happens when we take, inside the same class, higher order Darboux tranformations. (ii) We have got a class of triangular superpotentials leading also to Hermitian partners. At the same time these non-Hermitian superpotentials describe some symmetries of the generalized J-C Hamiltonians. We have also carried out the computations for just one example that, for $\ell = 0$, includes as a particular case the usual J-C Hamiltonian.

Acknowledgments

This work is partially supported by Spanish Ministerio de Educación y Ciencia under Project FIS2005-03989, Ministerio de Asuntos Exteriores (AECI grant 0000147625 of ŞK) and Junta de Castilla y León (Excellence Project VA013C05). The research of VH is partially supported by research grants from NSERC of Canada. The work reported in this paper was started when VH visited the University of Valladolid in 2005. She would like to thank the University of Valladolid for a sabbatical Fellowship. This paper was finished when JN and ŞK visited the University of Montreal in May 2006. They wish to thank the University of Montreal for hospitality and financial help. ŞK also acknowledges Department of Physics, Ankara University, Turkey and the warm hospitality at Department of Theoretical Physics, University of Valladolid, Spain.

References

[1] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
[2] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[3] Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21
[4] Fernández D J, Negro J and Nieto L M 2000 *Phys. Lett.* A **275** 338
[5] Negro J, Nieto L M and Rosas-Ortiz O 2000 *J. Phys. A: Math. Gen.* **33** 7207
[6] Andrianov A A, Cannata F, Ioffe M and Nishnianidsze D N 1997 *J. Phys. A: Math. Gen.* **30** 5037
[7] Goncharenko V M and Veselov A P 1998 *J. Phys. A: Math. Gen.* **31** 5315
[8] Samsonov B F and Pecheritsin A A 2004 *J. Phys. A: Math. Gen.* **37** 239
[9] Ioffe M V, Kuru Ş, Negro J and Nieto L M 2006 *J. Phys. A: Math. Gen.* **39** 6987
[10] Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89
[11] Bérubé-Lauzière Y, Hussin V and Nieto L M 1994 *Phys. Rev.* A **50** 1725
[12] Sukumar C V and Buck B 1984 *J. Phys. A: Math. Gen.* **17** 885
[13] Andreev V A and Lerner P B 1989 *Phys. Lett.* A **134** 507
[14] Alvarez-Moraga N and Hussin V 2003 *J. Phys. A: Math. Gen.* **36** 9479
[15] Samsonov B F and Negro J 2004 *J. Phys. A: Math. Gen.* **37** 10115
[16] Gendenstein L E 1983 *JETP Lett.* **38** 356
[17] Aleixo A N F, Balantekin A B and Cândido Ribeiro M A 2000 *J. Phys. A: Math. Gen.* **33** 3173
[18] Fernández D J, Negro J and del Olmo A 1996 *Ann. Phys., NY* **252** 386
[19] Flügge S 1971 *Practical Quantum Mechanics I* (Berlin: Springer)
[20] Abramowitz M and Stegun I A (ed) 1964 *Handbook of Mathematical Functions* (Washington: National Bureau of Standards)
[21] Bagrov V G and Samsonov B F 1997 *Phys. Part. Nucl.* **28** 374